ON CERTAIN QUESTIONS RELATED TO THE PROBLEM OF THE STABILITY OF UNSTEADY MOTION*

(O NEKOTORYKH VOPROSAKH, OTNOSIASHCHIKHSIA K ZADACHE OB USTOICHIVOSTI NEUSTANOVIVSHIKHSIA DVIZHENII)

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In modern engineering there arise new and increasingly more complex problems concerning the stability of motion. Looking at the past and anticipating the future, one can see that in order to keep up with technological progress it will be necessary to develop more and more precise methods for the investigation of these stability problems. The main difficulties in this direction are caused by the insufficient development of computation algorithms and of the procedures proposed already by Liapunov in his work "General problem of the stability of motion".

1. Some problems on the stability of motion. The possibility of the application of Liapunov's [1] method to the solution of important engineering problems of stability of motion was pointed out by me in my lectures on aircraft stability which I gave at Kazan' University in the early forties.

Liapunov's general theorems on stability (Section 16) apply, obviously, to the equations of perturbed motion

$$\frac{dx_s}{dt} = X_s(t,\ x_1, \ldots, x_n) \qquad (s = 1, \ldots, n)$$

in which the bounded, continuous, real functions X_s vanish when $x_1 = 0, \ldots, x_n = 0$, and satisfy the conditions for the existence of a single-valued solution in the region

$$t \geqslant t_0, \qquad x_1^2 + \ldots + x_n^2 \leqslant H$$

for arbitrary disturbances in this region.**

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* The work was published in a small number of copies in 1949.

** The notation is the same as that used in my textbook [2].

Liapunov accepted the following definition of stability.

If for an arbitrarily small, given positive number A there can be selected a positive number λ such that for all disturbances $x_{10}, \ldots, x_{n0}$, satisfying the condition

$$x_{10}^2 + \ldots + x_{n0}^2 \leqslant \lambda$$

and if for all t, $t > t_0$, the following inequality is valid

$$x_1^2 + \ldots + x_n^2 < A$$

then the undisturbed motion ($x_1 = 0, \ldots, x_n = 0$) is stable; in the opposite case, it is unstable.

In the proof of the theorem on stability (Section 16) which was given in the spirit of the epsilon proofs, Liapunov proposed a useful, practical method of finding (for a given number A, less than H) with the aid of the functions V and W, a positive number λ possessing the property specified in the definition of stability. This is a very important point.

In most engineering problems one is interested in satisfying the inequalities, appearing in the definition of stability, for given λ and A over a bounded time interval from the initial moment t_0 to some instant T. When the values of λ, A, t_0 and T are, however, given, then there arises the problem of the definition of the (λ, A, t_0, T)-stability in the large during a bounded interval of time.

Transforming, if necessary, the right hand sides of the differential equations of the disturbed motion in the problem of the (λ, A, t_0, T)-stability in the appropriate way in the regions*

$$x_1^2 + \ldots + x_n^2 < \lambda, \qquad A < x_1^2 + \ldots + x_n^2 \leqslant H$$

for every $t > t_0$, while in the region

$$\lambda \leqslant x_1^2 + \ldots + x_n^2 \leqslant A$$

for $t > T$, we can reduce the problem of the (λ, A, t_0, T)-stability to a more general stability problem of Liapunov with a certain additional restriction. This restriction is that the Liapunov functions of the

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* The possibility of transforming the given equations in the region $x_1^2 + \ldots + x_n^2 < \lambda$ by such a treatment of the given stability problem of Liapunov, is explained by the possibility of selecting (in the latter problem) the initial instant of time as any point in the interval (t_0, T).

transformed equations possess the properties specified by Liapunov for t greater than the given t_0, and that the number λ, obtained from A by Liapunov's method, be greater or equal to the given value for λ.

This circumstance makes the direct method of Liapunov quite valuable in the application to those applied problems of stability in the large during a bounded interval of time, for which there exists a general problem of Liapunov.

In Liapunov's definition of stability it is textually assumed that there are no disturbing forces in the sense, that the disturbed motion takes place under the action of those forces which were taken into consideration in the determination of the undisturbed motion. Liapunov gave, already in the problem of stability in the first approximation, the first examples of problems with disturbing forces. Obviously, not every problem of the (λ, A, t_0, T)-stability with disturbing forces can be covered by one of Liapunov's problems (for example, such a direct covering does not exist when $\lambda = 0$). The covering of the problem of the (λ, A, t_0, T)-stability with a Liapunov problem can be accomplished by various methods.

After these introductory remarks, I shall occupy myself in what follows with the problem of the stability of motion in the sense of Liapunov.

2. Theorem of instability for regular systems [3]. Everybody knows Liapunov's theorem: if the system of differential equations of the first approximation is regular and if all its characteristic numbers are positive, then the undisturbed motion is stable.

One can prove a theorem of instability that is a converse in a certain sense: if the system of the differential equations of the first approximation is regular, and if among its characteristic numbers there exists at least one negative number, then the undisturbed motion is not stable.

Let us consider the system of differential equations of the first approximation

$$\frac{dx_s}{dt} = p_{s1}x_1 + \dots + p_{sn}x_n \qquad (s = 1, \dots, n) \tag{1}$$

where the p_{sr} represent certain real continuous, bounded functions of t defined for all positive values of t. If this system is regular, then according to the definition of regular systems [1, Sect. 9] the sum $\lambda_1 + \dots + \lambda_n$ of the characteristic numbers λ_r of the normal system of its independent solutions

$$x_{1r}, \dots, x_{nr} \qquad (r = 1, \dots, n)$$

is equal to the negative of the characteristic number of the function

$$\exp - \int \sum p_{ss} dt$$

This is possible if the sum of the characteristic numbers of the functions

$$\exp \int \sum p_{ss} dt, \qquad \exp - \int \sum p_{ss} dt$$

is zero.

A system of n independent solutions is normal if the sum of the characteristic numbers of all remaining independent solutions attains its supremum [1, Sect. 8, Theorem IV].

We denote by Δ the determinant constructed of the functions x_{ij}, and by Δ_{ij} the cofactor (minor) of its element x_{ij}. It is well known that the functions

$$y_{sr} = \frac{\Delta_{sr}}{\Delta} \qquad (s = 1, \ldots, n) \tag{2}$$

satisfy, with r fixed, the system of linear differential equations associated with the problem of the system (1).

Let us denote by μ_r the characteristic number of the group of functions $y_{1r}, \ldots, y_{nr}$ of formula (2), which were determined by a normal system of independent solutions x_{ij} of the regular system (1). On the basis of general results of Liapunov on characteristic numbers [1, Sect. 6], we have the inequality $\mu_r \geqslant -\lambda_r$. From the obvious relation

$$\sum y_{sr} x_{sr} = 1$$

we deduce the inequality $\mu_r + \lambda_s \leqslant 0$. These inequalities lead to the relation

$$\mu_r + \lambda_r = 0 \qquad (r = 1, \ldots, n) \tag{3}$$

The system of differential equations that has been associated with (1) will, therefore, be regular also, and the system of functions y_{sr}, given by (2), will represent its normalized system of independent solutions.

Let us now consider the complete system of differential equations of the disturbed motion

$$\frac{dx_s}{dt} = p_{s1} x_1 + \ldots + p_{sn} x_n + X_s \qquad (s = 1, \ldots, n)$$

where, for all positive t, the X_s are holomophic functions of the quantities $x_1, \ldots, x_n$, at least for all those values of the latter which satisfy the condition

$$x_1^2 + \ldots + x_n^2 \leqslant A$$

where A is a constant distinct from zero and the coefficients in X_s are assumed to be determined by definite, continuous, bounded functions of t; the expansion of X_s begins with terms of at least the second order. Let us introduce the variables $z_1, \ldots, z_n$ in accordance with the formulas

$$z_r = \sum_s x_s y_{sr} y^{-\lambda_r t} \qquad (r = 1, \ldots, n)$$

From this it follows that the characteristic number of the group of functions $z_1, \ldots, z_n$ is smaller than the characteristic number of the group of functions $x_1, \ldots, x_n$, i.e.

$$\text{char. numb. } \{z_r\} > \text{char. numb. } \{x_s\} \tag{4}$$

We have

$$\frac{dz_r}{dt} = -(\lambda_r - \varepsilon) z_r + \sum X_s y_{sr} e^{-(\lambda_r - \varepsilon)t} \tag{5}$$

Let us now assume that among the characteristic numbers $\lambda_1, \ldots, \lambda_n$ there is at least one negative one. Let it be λ_1.

The instability of the undisturbed motion (relative to the variables $x_1, \ldots, x_n$) will be proved by the method of contradiction.

If the undisturbed motion is stable, then for an arbitrary given small positive number A there will exist such a positive number R that for arbitrary initial disturbances $x_{10}, \ldots, x_{n0}$ satisfying the inequality

$$x_{10}^2 + \ldots + x_{n0}^2 < R \tag{6}$$

the following inequality will hold for all positive t:

$$x_1^2 + \ldots + x_n^2 < A \tag{7}$$

Under this assumption it follows from the equation with $r = 1$ of system (5), that

$$z_1 = ce^{-\lambda_1 t} + e^{-\lambda_1 t} \int \sum_s X_s y_{s1} dt \qquad (c \text{ is some constant})$$

Since the functions X_s are assumed to be bounded for all positive values of t and for all $x_1, \ldots, x_n$ satisfying the condition (1), and since they are assumed to possess expansions beginning with the second degree terms in powers of the variables $x_1, \ldots, x_n$, it follows that in the selection of the initial values $x_{10}, \ldots, x_{n0}$, in accordance with the inequality (6), and for small enough R, we find that the characteristic number of the last term on the right hand side of the last equation is not negative and, hence, that the characteristic number of the function

z_1 is equal to λ_1 [1, Sect. 6, Lemma IV]. This and the relation (4) prove that the characteristic number of the group $x_1, \ldots, x_n$, which is equal to the smallest of the characteristic numbers of the functions $z_1, \ldots, z_n$, will be not larger than the characteristic number of z_1, i.e.

$$\text{char. numb. } \{x_s\} \leqslant \lambda_1 < 0$$

This statement contradicts the condition (7). We must, therefore, conclude that whatever the value of R may be, among the initial disturbances $x_{10}, \ldots, x_{n0}$ there exist some for which the inequality (7) ceases to hold for some positive values of t. Thus the theorem can be considered proved.

3. On some questions of the stability and instability for non-regular systems [4]. If the system of differential equations of the first approximation is not regular, then, indicating the sum of all characteristic numbers of the normalized system of its solutions by

$$s = \lambda_1 + \ldots + \lambda_n$$

and by μ the characteristic number of the function $1/\Delta$, we will have

$$s + \mu = -\sigma$$

where σ is some positive number.

In this case the characteristic number of the functions

$$y_{sr} = \frac{\Delta_{sr}}{\Delta}$$

will not be less than $-\lambda_r - \sigma$. For the sake of definiteness we shall asssume that the functions y_{sr} satisfy the conditions

$$\sum_s y_{sr}^2(0) = 1$$

Theorem of Liapunov. If the system of differential equations of the first approximation is not regular and if each of its characteristic numbers is greater than σ, then the undisturbed motion is stable.

Proof. Let us introduce the new variables

$$z_r = \sum_s x_s y_{sr} e^{-(\lambda_r - \epsilon)t}$$

where ϵ represents some positive number less than every one of the characteristic numbers λ_r and larger than σ. If the smallest of the characteristic numbers is denoted by λ_1, then $\lambda_1 > \epsilon > \sigma$.

The formulas for the inverse transformation will be

$$x_\alpha = \sum_r z_r x_{\alpha r} e^{-(\lambda_r - \epsilon)t}$$

The coefficients standing on the right hand sides of the linear forms in the variables z_r will be vanishing functions of time with characteristic numbers not less than ϵ. From the last formulas it follows that

$$\text{char.numb.}\{x_a\} \geqslant \text{char.numb.}\{z_s\} + \epsilon$$

Here the symbol $\{x_a\}$ stands for the system of functions $x_a(a = 1, \ldots, n)$.

Let us consider the positive definite quadratic form

$$2V = z_1^2 + \ldots + z_n^2$$

The total derivative with respect to time is, because of the given system of differential equations of the disturbed system,

$$\frac{dV}{dt} = -\sum_r (\lambda_r - \varepsilon) z_r^2 + R \qquad \left(R = \sum_{rs} z_r X_s y_{sr} e^{-(\lambda_r - \varepsilon)t}\right)$$

The function $R = R(t, z_1, \ldots, z_n)$, as a function of the new variables, has in its series expansion, in positive powers of the variables $z_1, \ldots, z_n$, coefficients which are vanishing functions of t with characteristic numbers not less than the positive number $\epsilon - \sigma$.

For every positive number η, no matter how small, one can find a region of sufficiently small numerical values $z_1, \ldots, z_n$ and a number T such that within this region and for all t greater than T the following inequality is valid

$$|R(t, z_1, \ldots, z_n)| < \eta(z_1^2 + \ldots + z_n^2)$$

This follows from the properties of $R(t, z_1, \ldots, z_n)$ as a function with vanishing coefficients and with an expansion that begins with terms of at least the third degree in z_s.

If η is chosen in accordance with the inequality $\lambda_1 - \epsilon > \eta$ with the indicated conditions, we shall have, for all $t > T$ and for all $z_1, \ldots, z_n$ in the specified region, the following relation

$$\frac{d}{dt}\sum_r z_r^2 \leqslant -2(\lambda_1 - \varepsilon - \eta)\sum_r z_r^2$$

Hence, if the initial values z_{s0} are chosen so that as t varies from t_0 to T, the values of the variables z_r lie in the indicated region, then for all $t > t_0$ we shall have

$$\sum_r z_r^2 \leqslant ce^{-2(\lambda_1 - \varepsilon - \eta)t}$$

Whence,

$$\text{char. numb. } \{z_r\} \geqslant \lambda_1 - \epsilon - \eta$$

and, hence

$$\text{char. numb. } \{x_s\} > \lambda_1 - \eta > 0$$

This proves the stability of the undisturbed motion relative to the variables $x_1, \ldots, x_n$, and also shows that every sufficiently close disturbed motion will tend asymptotically to the stable motion.

Theorem. If the system of differential equations of the first approximation is not regular, and if its smallest characteristic number is less than $-\sigma$, then the undisturbed motion is unstable.

Proof. We denote the smallest characteristic number of the equations of the first approximation by λ_1. By the hypotheses of the theorem $\lambda_1 + \sigma < 0$.

Let us consider the variables

$$z_r = \sum x_s y_{sr} e^{-(\lambda_r+\sigma)t}$$

whence,

$$\text{char. numb. } \{z_r\} > \text{char. numb. } \{x_s\}$$

We consider the equation

$$\frac{dz_1}{dt} = -(\lambda_1+\sigma) z_1 + \sum_s X_s y_{s1} e^{-(\lambda_1+\sigma)t}$$

The proof of the theorem will be made by contradiction. Let us assume that the undisturbed motion is stable under the given conditions. Then for every given positive number A, no matter how small, there will exist a positive number a such that, for initial disturbances $x_{10}, \ldots, x_{n0}$ satisfying the inequality

$$x_{10}^2 + \ldots + x_{n0}^2 \leqslant a$$

and for all t greater than t_0, the following inequality will hold:

$$x_1^2 + \ldots + x_n^2 < A$$

From the differential equation for z_1 it follows that

$$z_1 = ce^{-(\lambda_1+\sigma)t} + e^{-(\lambda_1+\sigma)t} \int_\infty^t \sum_s X_s y_{s1} dt$$

where c is some constant.

If the undisturbed motion is stable, and A is chosen smaller than the radius of the region of holomorphness of the functions X_s, then the functions X_s will be bounded for all values of $t > t_0$, provided, of course, that the initial disturbances are chosen in accordance with the inequality

$$x_{10}^2 + \ldots + x_{n0}^2 \leqslant a$$

The characteristic number of the system of functions y_{s1} is not smaller than $-(\lambda_1 + \sigma) > 0$. The limits in the integral were, therefore, chosen in accordance with known theorems of Liapunov on the characteristic number of an integral. If, without loss of generality, we let the initial moment of time be $t_0 = 0$, we obtain the relation

$$\sum x_{s0} y_{s1}(0) = c - \int_{\infty}^{0} \sum X_s y_{s1} dt$$

From this and the property of the holomorphic functions X_s, whose expansion of powers of $x_1, \ldots, x_n$ begin with terms of degree at least two, and from the property of the functions y_{s1}, which vanish with a positive characteristic number not less than $-(\lambda_1 + \sigma)$, we see that, for a numerically small enough A and for the largest a for the given A, the left hand side of the last equation will be a first order quantity, while the integral will be a quantity of order not less than the second. Hence, the constant c will be distinct from zero.

One can obtain this result more simply if in the preceding equations the variables x_i are expressed in terms of $t, x_{10}, \ldots, x_{10}$ under the assumption that the undisturbed motion is stable. In this case the integral will begin with a second order term in x_{s0} with a bounded coefficient. Hence, for a small enough a, one can find such values of x_{s0} that c will be different from zero.

From the formula for z_1 we deduce that

$$\text{char. numb. } z_1 = \lambda_1 + \sigma < 0$$

and, hence,

$$\text{char. numb. } \{x_n\} \leqslant \lambda_1 + \sigma < 0$$

which contradicts the hypothesis on the stability of the non-disturbed motion. This proves the theorem.

4. On the sign of the smallest characteristic number. Among the problems concerning the stability of motion, the one dealing with the sign of the smallest characteristic number of a system of linear differential equations is of special interest. Its significance is illuminated by the preceding theorems.

In the general formulation the problem of the sign of the smallest characteristic number has not been solved [5] in a form which would lend itself effectively for computations, even for equations with constant coefficients. For the latter equations the determination of the characteristic numbers does not present a problem.

The reasons why the various methods for the determination of the characteristic number in the general formulation of a problem yield results which are quite ineffective for computations are not clear at the present time; furthermore, it is not clear what minimal properties of the coefficients p_{rs} are to be dropped or even to be forbidden in order to obtain an effective solution of the problem. This matter could be resolved with the correct general statement of the problem of the characteristic number.

5. On the upper and lower bounds of characteristic numbers. The first results on the more or less precise upper and lower bounds of characteristic numbers for the linear differential equations (1) were established by Liapunov in the proof of the theorem that every non-trivial solution of the system of differential equations (1) has a finite characteristic number.

Proof. Let us consider a real solution in which the x_s are real functions of t. We introduce the new variables

$$z_s = x_s e^{\lambda t}$$

where λ denotes some real constant.

Then the given equations (1) will be transformed into the following ones:

$$\frac{dz_s}{dt} = p_{s1}z_1 + \ldots + (p_{ss} + \lambda)\, z_s + \ldots + p_{sn}x_n \quad (s = 1, \ldots, n)$$

from which we derive

$$\frac{1}{2}\,\frac{d}{dt}\sum_s z_s^2 = \sum_{sr} (p_{sn} + \delta_{sr}\lambda)\, z_s z_r$$

The second part of this equation represents some real quadratic form in z, with coefficients depending on λ, and t. Because of the assumed boundedness of the function p_{sr} one can find such values $\lambda = \lambda'$, for which all principal diagonal minors of the discriminant

$$\left\| \frac{p_{sr} + p_{rs}}{2} + \delta_{sr}\lambda \right\|$$

will be positive for all values of t under consideration. For such a value of λ', the quadratic form will be positive-definite.

One can also find such values $\lambda = \lambda_1$ for which the principal diagonal minors will alternate in sign beginning with a certain negative one $p_{11} + \lambda_1$. For such a λ_1, the quadratic form standing on the right hand side will be negative-definite.

From this it follows that for every $\lambda = \lambda' + \epsilon/2$, where ϵ is an arbitrary positive number, we have the equation

$$\frac{d}{dt}\sum z_s^2 > \varepsilon \sum_s z_s^2$$

from which we obtain by integration the inequality

$$\sum_s z_s^2 > ce^{\varepsilon t}$$

for all t under consideration. Here, c stands for some positive constant which does not exceed $\Sigma z_{s0}^2 e^{-\epsilon t_0}$, where the z_{s0} are the initial values of the variables z_s corresponding to the initial time t.

If $\lambda = \lambda_1 - \epsilon/2$, we have

$$\frac{d}{dt}\sum_s z_s^2 < -\varepsilon \sum_s z_s^2$$

whence,

$$\sum_s z_s^2 < c'e^{-\varepsilon t}$$

for all t under consideration. Here, c^1 stands for a positive constant not less than $e^{\epsilon t_0}\, \Sigma z_{s0}^2$.

Hence, for $\lambda = \lambda_1 + \epsilon/2$, where ϵ is an arbitrary positive number, one can find among the functions z_s at least one unbounded one; while for $\lambda = \lambda' - \epsilon/2$, all the functions z_s will be non-decreasing functions. Thus, the smallest characteristic number of the functions x_s of the real, non-trivial solution of the system (1) under consideration will not be less than λ_1 and not greater than λ'.

Consequence [2]. If the coefficients p_{sr} of the differential equations (1) are such that the principal diagonal minors of the determinant

$$\| p_{sr} + p_{rs} \|$$

alternate in sign, whereby p_{11} is negative for all values of t greater than some constant t_0, then the characteristic numbers of the particular solutions of such a system are all positive.

The bound for the characteristic numbers determined by Liapunov can be improved. Indeed, in accordance with the given equations we have

$$\frac{1}{2}\frac{d}{dt}\sum_s x_s^2 = \sum_{sr} p_{sr} x_s x_r$$

If the right hand side of this equation is symmetrized and if the symmetric quadratic form is reduced to the sum of squares by means of a linear orthogonal transformation, then one can obtain the known inequalities

$$\alpha \sum_s x_s^2 \leqslant \sum_{s,r} p_{sr} x_s x_r \leqslant \beta \sum_s x_s^2$$

where α and β denote the smallest and largest roots, respectively, of the equation

$$\left\| \frac{p_{sr} + p_{rs}}{2} - \delta_{sr} \varkappa \right\| = 0$$

Hence,

$$2\alpha \sum_s x_s^2 \leqslant \frac{d}{dt} \sum x_s^2 \leqslant 2\beta \sum_s x_s^2$$

or

$$c \exp \left(2 \int \alpha dt\right) \leqslant \sum_s x_s^2 \leqslant c \exp \left(2 \int \beta dt\right) \qquad (c = x^2_{10} + \ldots + x^2_{n0})$$

where the integrals are taken with the limits t_0 to t.

On the basis of this we must conclude that the characteristic number of the system of functions $x_1, \ldots, x_n$ satisfying the given linear equations, is not larger than the characteristic number of the expression $\exp \int \alpha dt$ and is not less than the characteristic number of the function $\exp \int \beta dt$, i.e.

char. numb. $\exp \int \alpha dt \geqslant$ char. numb. $\{ x_s \} \geqslant$ char. numb. $\exp \int \alpha dt$

Consequence. The smallest characteristic number of the solutions of the given equations will be positive if the characteristic number of the function

$$\exp \int \beta dt$$

is positive, where β stands for the largest root of the equation

$$\left\| \frac{p_{sr} + p_{rs}}{2} - \delta_{sr} \varkappa \right\| = 0$$

6. The coefficients p_{sr} tend towards the definite limits c_{sr} [8].

Theorem. If with the unbounded increase of t, the coefficients p_{rs} tend to the definite limits c_{sr}, the smallest characteristic number of the equations (1) coincides with the smallest characteristic number of the limiting system

$$\frac{dx_s}{dt} = c_{s1} x_1 + \ldots + c_{sn} x_n \qquad (s = 1, \ldots, n) \tag{8}$$

Proof. We make the substitution

$$z_s = x_s e^{\eta t} \quad (s = 1, \ldots, n)$$

where η is some constant. The given equations (1) will be transformed into the system

$$\frac{dz_s}{dt} = p_{s1}z_1 + \ldots + (p_{ss} + \eta) z_s + \ldots + p_{sn}z_n \tag{9}$$

while the limiting system (8) will go over into the next system, which is the limiting system for Equation (9),

$$\frac{dz_s}{dt} = c_{s1}z_1 + \ldots + (c_{ss} + \eta) z_s + \ldots + c_{sn}z_n \tag{10}$$

The roots of the characteristic equation of the system (10), $|| c_{sr} - \delta_{sr}(\chi - \eta) || = 0$, we will indicate by $\chi_1, \ldots, \chi_n$.

If there exist no non-negative integers $m_1, \ldots, m_n$, whose sum is 2, for which the expression

$$m_1 \varkappa_1 + \ldots + m_n \varkappa_n$$

vanishes, then there will exist a quadratic form W with positive coefficients satisfying the equation

$$\sum \frac{\partial W}{\partial z_s} [c_{s1}z_1 + \ldots + (c_{ss} + \eta) z_s + \ldots + c_{sn}z_n] = z_1^2 + \ldots + z_n^2$$

The form W will be negative-definite if the real parts of all the roots χ_s are negative; this form W will take on positive values for certain values of the variable z if there exists at least one root (among the roots $\chi_1, \ldots, \chi_n$ with a positive real part).

In view of Equation (9), the total derivative of such a function W with respect to t will have the form

$$\frac{dW}{dt} = z_1^2 + \ldots + z_n^2 + \sum (p_{sr} - c_{sr}) \frac{\partial W}{\partial z_s} z_r$$

Since the function W is a quadratic form with constant coefficients, one can find a number $\epsilon > 0$ such that whenever

$$| p_{sr} - c_{sr} | < \varepsilon$$

the right hand side of the last equation will represent a positive quadratic form in the variables z.

The coefficients p_{sr}, however, tend to the limit c_{sr} as t increases indefinitely. Hence, for every positive ϵ, no matter how small, there exists a T such that for all $t > T$ the absolute values of the differences $p_{sr} - c_{sr}$ will be less than ϵ, and, hence, for all t greater than T, the derivative dW/dt will be a positive-definite function.

This leads us to conclude on the basis of the general stability theorems of Liapunov, that if there do not exist non-negative numbers

$m_1, \ldots, m_n$ whose sum is 2, for which the expression

$$m_1\varkappa_1 + \ldots + m_n\varkappa_n$$

vanishes, and if the smallest characteristic number (taken with opposite sign of the largest real part of the roots χ_s) of the system (10) is positive, then the undisturbed motion of the system (9) is asymptotically stable; while, if the smallest characteristic number of the system (10) is negative, the undisturbed motion of the system (9) is unstable.

From this we may conclude that the smallest characteristic numbers of the set of functions $z_1, \ldots, z_n$ when the $x_1, \ldots, x_n$ are solutions of the given equations (1), as well as those of the limiting system (8), can be equal to zero only for one definite value of the constant η. This proves the theorem.

7. The coefficients p_{sr} have bounded oscillations [2]. If the coefficients of the linear equations (1) have the form

$$p_{sr} = c_{sr} + \varepsilon f_{sr}$$

where ϵ is a parameter, the c_{sr} are independent of ϵ, and the f_{sr} are bounded real functions of t, then the given equations (1) include as particular cases the equations with the constant coefficients c_{sr}:

$$\frac{dx_s}{dt} = c_{s1}x_1 + \ldots + c_{sn}x_n \qquad (s = 1, \ldots, n)$$

Let us assume that the roots λ_s of the characteristic equation

$$\| c_{sr} - \delta_{sr}\,\lambda \| = 0$$

satisfy the condition $m_1\lambda_1 + \ldots + m_n\lambda_n \neq 0$ for arbitrary non-negative integers whose sum is 2. We consider the quadratic form ($a_{rs} = a_{sr}$) with constant coefficients

$$2V = \sum a_{rs}x_r x_s$$

determined by the equations

$$\sum_s (c_{s1}x_1 + \ldots + c_{sn}x_n)\frac{\partial V}{\partial x_s} = -x_1^2 - \ldots - x_n^2$$

The total derivative of V with respect to t can be written, in view of the last equation and of the given equations (1), as

$$V' = -x_1^2 - \ldots - x_n^2 + \varepsilon\Sigma\,(f_{s1}x_1 + \ldots + f_{sn}x_n)\frac{\partial V}{\partial x_s}$$

For small enough $|\epsilon|$ and for a positive μ less than 1, the form $-V' - (x_1^2 + \ldots + x_n^2)$ can be made positive for all t and for arbitrary values of the variables. For such a value of ϵ, the asymptotic stability or instability of the undisturbed motion ($x_1 = 0, \ldots, x_n = 0$) of the equation with the constant coefficients c_{sr} will correspond to the stability and unstability of the given system (1); the value of ϵ, for which such a correspondence will unconditionally exist, is determined by n inequalities for all $t > t_0$:

$$(-1)^r \begin{vmatrix} h_{11} \dots h_{1r} \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ h_{r1} \dots h_{rn} \end{vmatrix} > \theta > 0 \qquad (r = 1, \dots, n)$$

where

$$h_{rs} = -\delta_{sr} + \frac{\varepsilon}{2}\sum_i (\alpha_{ri} f_{is} + \alpha_{si} f_{ir}) = h_{sr}$$

For the case of stability which is of interest to us, one can sharpen the estimate given in the preceding discussion.

For this purpose we consider the extremal values of V' on the surface $V - e = 0$. We make use of Lagrange's method.

We have the equations for the extremum

$$\frac{\partial V'}{\partial x_s} = \lambda \frac{\partial V}{\partial x_s}$$

Hence, for the extremal positions we have

$$V' = \lambda V$$

where λ is a root of the equation

$$\| 2h_{rs} - \lambda a_{rs} \| = 0$$

Let λ_1 be the smallest, λ' the largest root of this equation; then, if V is positive-definite,

$$\lambda_1 V \leqslant V' \leqslant \lambda' V$$

Hence,

$$V_0 \exp \int_{t_0}^{t} \lambda_1 dt \leqslant V \leqslant V_0 \exp \int_{t_0}^{t} \lambda' dt$$

From this we deduce that for a positive-definite V, the upper and lower bounds of the characteristic numbers of the solutions are determined by the characteristic numbers of the expressions

$$\exp\left(\frac{1}{2}\int_{t_0}^{t} \lambda' \, dt\right), \qquad \exp\left(\frac{1}{2}\int_{t_0}^{t} \lambda_1 \, dt\right)$$

(*Note*. The last inequality yields the possibility of solving the problem of the (λ, A, t_0, T)-stability. Let V be a positive-definite quadratic form. Let us assume that c is the exact maximum of V on the sphere $x_1^2 + \dots + x_n^2 = \lambda$, and that C is the exact lower bound of V on the sphere A. Then

$$|V_0| \leqslant c$$

and, hence, in order to have (λ, A, t_0, T)-stability it is sufficient to satisfy the inequality

$$C < c \exp \int^{t} \lambda' dt$$

for all t on the interval (t_0, T)).

8. Parametric considerations [8]. The equation

$$\Delta(\lambda) = \| p_{sr} - \delta_{sr} \lambda \| = 0$$

will have n roots $\lambda_1, \ldots, \lambda_n$ for each value of t. These roots will change with a change in t.

If for every positive t there exist no non-negative integers $m_1, \ldots, m_n$, whose sum is 2, for which the expression

$$m_1\lambda_1 + \ldots + m_n\lambda_n$$

vanishes, then, for such t, there will exist a quadratic form

$$V = \sum_{sr} a_{sr} x_s x_r \qquad (a_{sr} = a_{rs})$$

with bounded coefficients depending on t. This V will satisfy the first order partial differential equation

$$\sum \frac{\partial V}{\partial x_s} (p_{s1} x_1 + \ldots + p_{sn} x_n) = - x_1^2 - \ldots - x_n^2$$

where t plays the role of a parameter.

The form V will be positive if the real parts of all the roots λ_s are negative; for certain values of the variables x_s, the form will take on negative values if there exists at least one root, among the $\lambda_1, \ldots, \lambda_n$, with a positive real part. The total derivative of V with respect to time can be written in the form

$$\frac{dV}{dt} = - x_1^2 - \ldots - x_n^2 + \frac{\partial V}{\partial t}$$

in view of Equations (1).

The discriminant of the quadratic form, standing on the right hand side of the last equation, is

$$D = \| a_{sr}' - \delta_{sr} \| \qquad \left(a_{sr}' = \frac{da_{sr}}{dt} \right)$$

Suppose that for all positive values of t, the derivatives a_{rs}' are bounded, and that all principal diagonal minors $D_1, \ldots, D_n$ of the discriminant D satisfy the inequalities $(-1)^r D_r > 0$, and that their absolute values are not less than some positive number. In this case the derivative will be (in accordance with a known criterion of Sylvester) a negative-definite quadratic form in the variables $x_1, \ldots, x_n$. Under these conditions, if V is a positive-definite quadratic form, the undisturbed motion will be stable; if V in addition does have an infinitesimally small upper bound, then the stability of the undisturbed motion will be an asymptotic stability. If, however, the form V admits an infinitesimally small upper bound and can take on negative values, then the undisturbed motion is unstable.

The parametric consideration can be useful for practical purposes when the p_{sr} change slowly with time.

For the case when V is a definitely positive function, the obtained results can be made more precise. Indeed, let us consider the problem on the extremal values

$$V' = \Sigma\,(a_{rs}' - \delta_{rs})\,x_r\,x_s \quad \text{on the surface } V = c$$

The equations of the extremal problem

$$\frac{\partial V'}{\partial x_s} = \lambda\,\frac{\partial V}{\partial x_s} \qquad (s = 1, \ldots, n)$$

for the extremal values of V' yield

$$V' = \lambda V$$

where λ is a root of the equation

$$\| a_{rs}' - \delta_{rs} - \lambda a_{rs} \| = 0$$

If λ is the smallest, and λ' the largest root of this equation, and if V is positive-definite we will have

$$\lambda_1 V \leqslant V' \leqslant \lambda' V$$

or

$$V_0 \exp\Big(\int_{t_0}^{t} \lambda_1\, dt\Big) \leqslant V \leqslant V_0 \exp\Big(\int_{t_0}^{t} \lambda'\, dt\Big)$$

This inequality makes it possible to determine the bounds for characteristic numbers $\lambda_1, \ldots, \lambda_n$, and it also can be directly useful in the consideration of the problem on the (λ, A, t_0, T)-stability.

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